Inverse Problems **24** (2008) 015009 (5pp) doi:10.1088/0266-5611/24/1/015009

How general are general source conditions?

Peter Mathé[1] **and Bernd Hofmann**[2]

[1] Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany
[2] Department of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, Germany

E-mail: mathe@wias-berlin.de and hofmannb@mathematik.tu-chemnitz.de

Received 25 September 2007, in final form 12 November 2007
Published 9 January 2008
Online at stacks.iop.org/IP/24/015009

Abstract
Error analysis of regularization methods in Hilbert spaces is based on smoothness assumptions in terms of source conditions. In the traditional setup, i.e. when smoothness is in a power scale, we see that not all elements in the underlying Hilbert space possess some smoothness with this scale. Our main result asserts that this can be overcome when turning to general source conditions defined in terms of index functions. We conclude with some consequences.

The issue of *smoothness* is essential when treating ill-posed inverse problems. On the one hand, smoothing properties of the forward operator are responsible for the ill-posedness, but linear and nonlinear ill-posed problems become conditionally well posed when restricted to appropriate classes of solutions with imposed smoothness. On the other hand, theoretical rates of reconstruction by regularization methods depend on the interplay between the smoothing properties of the forward operator and the smoothness of the solution.

Regularization theory in a Hilbert space setting is well developed and proves to be particularly simple when the character of solution smoothness is immediately tied to the forward operator for linear inverse problems, or to its linearization in the nonlinear case.

To avoid technical difficulties in our subsequent discussion, we shall focus on linear equations

$$Ax = y, \tag{1}$$

where the forward operator $A : X \to Y$ acts injectively between the Hilbert spaces X and Y, with norms $\|\cdot\|$ and inner products $\langle\cdot,\cdot\rangle$, and has a nonclosed range $\mathcal{R}(A)$. Then the related nonnegative selfadjoint operator $H := A^*A$ is injective and has nonclosed but dense range $\mathcal{R}(H) \subset X$. We mention that in this case zero must be an accumulation point of the spectrum of the operator H.

Since the early error estimates for regularization methods of such operator equations in Hilbert spaces smoothness has been measured in terms of *source conditions* (source-wise

0266-5611/08/015009+05$30.00 © 2008 IOP Publishing Ltd Printed in the UK

representations). The first appearances are probably in [11] by Lavrentev and in [15] by Morozov. We also mention the influential contributions by Vainikko [18], Groetsch [5], Baumeister [1], Louis [12], and the more recent fundamental monograph by Engl *et al* [4]. Precisely, it is assumed that there are two values $0 < p < \infty$ and $0 < R < \infty$, for which

$$x \in H_p(R) := \{x : x = H^p v, \|v\| \leqslant R\}. \tag{2}$$

The representability of x in such form (2) can be rephrased by saying that $x \in \mathcal{R}(H^p)$. If the operator H were such that $\mathcal{R}(H) = X$, then H would be boundedly invertible and so would be H^p for all $p > 0$, thus $\mathcal{R}(H^p) = X$, and each x would obey a representation in terms of source sets (2) for every $p > 0$. Hence the case of operators H with closed range is not of interest.

Now we are going to establish the first result, which is not surprising.

Proposition. *For each injective, bounded, selfadjoint and nonnegative linear operator $H: X \to X$ with nonclosed range there is an element $x \in X$ which does not belong to the source set $H_p(R)$ for any $0 < p < \infty$ and $0 < R < \infty$.*

Proof. We fix some $\mu > 0$ and let $\varphi(t) := \log^{-\mu}(a/t)\,(0 < t \leqslant \|H\| < a)$. The assumed properties of H carry over to the corresponding operator $\varphi(H)$. We claim that $\mathcal{R}(\varphi(H)) \not\subset \mathcal{R}(H^p)$, for any $p > 0$. Indeed, if for some $p > 0$ the inclusion $\mathcal{R}(\varphi(H)) \subset \mathcal{R}(H^p)$ were true, then by the reasoning below the operator $H^{-p}\varphi(H)$ must be bounded on X, which in turn implies that the function $\varphi(t)/t^p (0 < t \leqslant a)$ also must be bounded, since zero is an accumulation point of the spectrum of H. However, the quotient of $\varphi(t)$ and t^p tends to infinity as $t \to 0$. Hence, there is some $v \in X$ for which $x := \varphi(H)v$ does not belong to the range $\mathcal{R}(H^p)$.

It remains to show that $\mathcal{R}(\varphi(H)) \subset \mathcal{R}(H^p)$ implies the boundedness of the operator $H^{-p}\varphi(H)$. To this end we recall one result from ([2], proposition 2.1(a)): If for some pair $S, T: X \to X$ of bounded selfadjoint linear operators the range inclusion $\mathcal{R}(S) \subset \mathcal{R}(T)$ is valid, and if T is injective, then necessarily the mapping $T^{-1}S: X \to X$ is bounded. This immediately yields the required boundedness, and hence completes the proof. □

The function $\varphi(t) := \log^{-\mu}(a/t)(0 < t \leqslant \|H\| < a)$ used in the proof leads to study more *general source conditions* of the form

$$x \in H_\psi(R) := \{x : x = \psi(H)\, v, \|v\| \leqslant R\}, \tag{3}$$

where the function $\psi: [0, \|H\|] \to (0, \infty)$ is an *index function* in the sense of [8, 14], i.e., it is continuous, strictly increasing, and obeys $\psi(0) = 0$. Such general source conditions naturally occur when treating *severely ill-posed* problems. We mention [13] for a first analysis in this context, and [9], where logarithmic source conditions like above were used.

The analysis of ill-posed equations under general source conditions (3) was pioneered independently by Hegland [6, 7] and Tautenhahn [17]. A more systematic account was started in [14] and continued in subsequent papers by these authors. As a function of the regularization parameter the present authors analyzed the noiseless residual errors (called profile functions) of regularization methods under general source conditions in [8] implying most relevant assertions on convergence rates.

However, one principal problem was left untouched: How general are general source conditions? Complementing the above proposition we now ask, given an operator $H: X \to X$, whether every $x \in X$ obeys a general source condition for some index function ψ? Apparently, no authors discussed this principal question so far. In the theorem below we shall give an affirmative answer to this question in case that A, and hence H, are *compact* operators.

Returning to the argument in the proof of the proposition we also cannot expect an affirmative result to hold if we replace the power scale in (2) by other scales, as e.g. *logarithmic scales* with functions $\varphi(t) = \log^{-\mu}(a/t)(0 < t \leqslant \|H\| < a)$ with μ ranging in $(0, \infty)$. In fact, then we could take double logarithmic functions for which failure of representability could be assured. The same would hold true for poly-logs, and so on. Thus some essentially different reasoning must be used, and we turn to the main result in this note.

Theorem. *Let $H: X \to X$ be a compact, injective, selfadjoint and non-negative linear operator. For every $x \in X$ and $\varepsilon > 0$ there is an index function ψ such that $x \in H_\psi(R)$ for $R = (1+\varepsilon)\|x\|$.*

Proof. Let $\{s_j, u_j\}_{j=1}^\infty$ be the eigensystem of the compact operator H, where $\{u_j\}_{j=1}^\infty \subset X$ is a complete orthonormal system, and $\|H\| = s_1 \geqslant s_2 \geqslant \ldots > 0$ denote the corresponding eigenvalues of finite multiplicities, arranged in non-increasing order. Under the above assumptions, zero is the only accumulation point of the set of eigenvalues $S(H) := \{s_j\}_{j=1}^\infty$, which coincides with the spectrum of H, thus $\mathbb{N}$ is disjointly decomposed as $\mathbb{N} := \bigcup_{\lambda \in S(H)} B_\lambda$ into finite sets B_λ of positive integers, for which $s_j = \lambda$, $j \in B_\lambda$. We let $n_\lambda := \min_{j \in B_\lambda} j$ for all $\lambda \in S(H)$.

We denote the Fourier coefficients of x by $\xi_j := \langle x, u_j \rangle$, $j \in \mathbb{N}$. Plainly, $\|x\|^2 = \sum_{j=1}^\infty |\xi_j|^2$. The proof is based on the following technical assertion, which is well known (see e.g. [16], section 8.6.4): For every $\varepsilon > 0$ there is a sequence $1 \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \ldots > 0$, such that $\lim_{j\to\infty} \sigma_j = 0$ and

$$\sum_{j=1}^\infty \frac{|\xi_j|^2}{\sigma_j^2} \leqslant (1+\varepsilon)\|x\|^2.$$

The sequence $\{\sigma_j\}_{j=1}^\infty$, may not be strictly decreasing, and we auxiliarily assign the strictly decreasing sequence

$$\mu_j^2 := \frac{1+\varepsilon/j}{1+\varepsilon}\sigma_j^2, \qquad j \in \mathbb{N},$$

which allows for an estimate

$$\sum_{j=1}^\infty \frac{|\xi_j|^2}{\mu_j^2} \leqslant (1+\varepsilon)^2\|x\|^2.$$

On the spectrum $S(H)$ we define the strictly increasing function

$$\tilde{\psi}(\lambda) := \mu_{n_\lambda}, \qquad \lambda \in S(H). \tag{4}$$

By construction the function $\tilde{\psi}$ obeys $\lim_{\lambda\to 0} \tilde{\psi}(\lambda) = 0$. By linear interpolation it may be extended continuously and strictly increasing to an index function ψ. Finally,

$$\begin{aligned}
\sum_{j=1}^\infty \frac{|\xi_j|^2}{\psi^2(s_j)} &= \sum_{\lambda\in S(H)} \sum_{j\in B_\lambda} \frac{|\xi_j|^2}{\psi^2(s_j)} \\
&= \sum_{\lambda\in S(H)} \sum_{j\in B_\lambda} \frac{|\xi_j|^2}{\mu_{n_\lambda}^2} \\
&\leqslant \sum_{\lambda\in S(H)} \sum_{j\in B_\lambda} \frac{|\xi_j|^2}{\mu_j^2} \leqslant (1+\varepsilon)^2\|x\|^2.
\end{aligned}$$

Thus we let $v := \sum_{j=1}^\infty \frac{\xi_j}{\psi(s_j)} u_j$. By the above calculations v is a well-defined element in X, with $\|v\| \leqslant (1+\varepsilon)\|x\|$, and $x = \psi(H)v$. □

The above result may have various implications, and we mention a few of those.

Corollary 1. *Given any $x \in X$ and a compact, injective, selfadjoint, non-negative linear operator $H: X \to X$, there is no maximal smoothness with respect to H under all index functions ψ with $x \in H_\psi(R)$ for some $R > 0$.*

Proof. Indeed, we could iterate the argument of the theorem: For $x = \psi_1(H)v_1$, $\|v_1\| \leqslant R_1$, we find an index function ψ_2 with $v_1 = \psi_2(H)v_2$ and $\|v_2\| \leqslant R_2$. But then $x \in \mathcal{R}(\psi_1(H)\psi_2(H))$, where $\psi_1(t)\psi_2(t)$ is an index function with higher decay rate to zero than $\psi_1(t)$ as $t \to 0$. □

Even though, due to Corollary 1, no element has maximal smoothness with respect to H, it still may have *unlimited* or *limited* smoothness. In the latter case the corollary asserts that no limit is attainable.

Other consequences of the theorem are from the theory of linear regularization methods, and we refer to ([4], chapter 4) for details on this subject. The following result strengthens the assertion of the theorem and has implications for bounding the residual errors in regularization at the solution (the profile functions), where we refer to [8] for details.

Corollary 2. *Given any $x \in X$ and a compact, injective, selfadjoint, non-negative linear operator $H: X \to X$, there are a concave index function ψ and a constant $R > 0$ such that $x \in H_\psi(R)$. Consequently, Tikhonov regularization yields a profile function $\|\alpha\,(A^*A + \alpha I)^{-1}x\| \leqslant R\psi(\alpha)$ at x.*

Proof. Using the theorem we find an index function, say ψ_0, and $R_0 > 0$ such that $x \in H_{\psi_0}(R_0)$. Now we continue in several steps. First we let

$$\omega_{\psi_0}(\delta) := \sup\{|\psi_0(s) - \psi_0(t)|, 0 \leqslant s, t \leqslant a, |s-t| \leqslant \delta\}, \qquad \delta > 0,$$

denote the *modulus of continuity* of ψ_0. This is a continuous, non-decreasing function which obeys $\lim_{\delta\to 0}\omega_{\psi_0}(\delta) = 0$. Plainly, $\psi_0(t) \leqslant \omega_{\psi_0}(t), 0 < t \leqslant a$. Moreover, for every modulus of continuity ω, see e.g. ([10], lemma 6.1.4), there is a *concave* function $\tilde{\omega}$, with $\tilde{\omega}(t) \leqslant \omega(t) \leqslant 2\tilde{\omega}(t), t > 0$, i.e. $2\tilde{\omega}_{\psi_0}$ is a concave majorant of ψ_0. In a final step we ensure strict monotonicity by letting

$$\psi(t) := \tilde{\omega}_{\psi_0}(t) + t/a, \qquad 0 \leqslant t \leqslant a,$$

which is a concave index function. We claim that $x \in H_\psi(R)$ for $R = 2R_0$. Precisely, for the eigensystem $\{s_j, u_j\}_{j=1}^\infty$ of the compact operator H and $\xi_j := \langle x, u_j\rangle$, $j \in \mathbb{N}$, we have to show that

$$\sum_{j=1}^\infty \frac{|\xi_j|^2}{\psi^2(s_j)} \leqslant R^2 \qquad \text{for} \quad R = 2R_0, \tag{5}$$

taking into account that $\sum_{j=1}^\infty \frac{|\xi_j|^2}{\psi_0^2(s_j)} \leqslant R_0^2$. Now the following inequality chain holds true:

$$\sum_{j=1}^\infty \frac{|\xi_j|^2}{\psi^2(s_j)} \leqslant \sum_{j=1}^\infty \frac{|\xi_j|^2}{\tilde{\omega}_{\psi_0}^2(s_j)} \leqslant 4\sum_{j=1}^\infty \frac{|\xi_j|^2}{(2\tilde{\omega}_{\psi_0}(s_j))^2} \leqslant 4\sum_{j=1}^\infty \frac{|\xi_j|^2}{\omega_{\psi_0}^2(s_j)} \leqslant 4\sum_{j=1}^\infty \frac{|\xi_j|^2}{\psi_0^2(s_j)},$$

which yields the estimate (5).

The remaining assertion follows from known results (see e.g. [14], lemma 2), or the more recent ([2], proposition 3.3). □

The existence of a profile function, as exemplarily shown in Corollary 2 for the Tikhonov regularization, is important for the *a posteriori* parameter choice with the Lepskiĭ balancing principle, and we refer to the recent paper [3] as one example. Other consequences of the above results still need to be explored.

Note added in proof. Sergei V Pereverzev (Linz) pointed out to us one important consequence of Corollary 1: Given noisy data it is impossible to obtain order optimal regularization by any *a priori* parameter choice rule, which is based on smoothness. This strongly supports the use of *a posteriori* rules for regularization.

References

[1] Baumeister J 1987 *Stable Solution of Inverse Problems* (Braunschweig: Vieweg)

[2] Böttcher A, Hofmann B, Tautenhahn U and Yamamoto M 2006 Convergence rates for Tikhonov regularization from different kinds of smoothness conditions *Appl. Anal.* **85** 555–78

[3] Cao H and Pereverzev S V 2007 The balancing principle for the regularization of elliptic Cauchy problems *Inverse Problems* **23** 1943–61

[4] Engl H W, Hanke M and Neubauer A 1996 *Regularization of Inverse Problems* (Dordrecht: Kluwer)

[5] Groetsch C W 1984 *The Theory of Tikhonov Regularization for Fredholm Integral Equations of the First Kind* (Boston: Pitman)

[6] Hegland M 1992 An optimal order regularization method which does not use additional smoothness assumptions *SIAM J. Numer. Anal.* **29** 1446–61

[7] Hegland M 1995 Variable Hilbert scales and their interpolation inequalities with applications to Tikhonov regularization *Appl. Anal.* **59** 207–23

[8] Hofmann B and Mathé P 2007 Analysis of profile functions for general linear regularization methods *SIAM J. Numer. Anal.* **45** 1122–41

[9] Hohage T 1997 Logarithmic convergence rates of the iteratively regularized Gauss-Newton method for an inverse potential and an inverse scattering problem *Inverse Problems* **13** 1279–99

[10] Korneĭchuk N 1991 Exact constants in approximation theory *Encyclopedia of Mathematics and its Applications* vol 38 ed K Ivanov (Cambridge: Cambridge University Press) (Translated from Russian)

[11] Lavrenev M M 1962 O nekotorykh nekorrektnykh zadachakh matematicheskoi fiziki *Izdat. Sibirsk. Otdel. Akad. Nauk SSSR* Novosibirsk

[12] Louis A K 1989 *Inverse und Schlecht Gestellte Probleme* (Stuttgart: Teubner)

[13] Mair B A 1994 Tikhonov regularization for finitely and infinitely smoothing operators *SIAM J. Math. Anal.* **25** 135–47

[14] Mathé P and Pereverzev S V 2003 Geometry of linear ill-posed problems in variable Hilbert scales *Inverse Problems* **19** 789–803

[15] Morozov V A 1968 O printsipe newyaski pri reshenii operatornykh uravneniĭ metodom regulyarizatsii *Ž. Vyčisl. Mat. i Mat. Fiz.* **8** 295–309

[16] Pietsch A 1980 *Operator Ideals* (Amsterdam: North-Holland)

[17] Tautenhahn U 1998 Optimality for ill-posed problems under general source conditions *Numer. Funct. Anal. Optim.* **19** 377–98

[18] Vaĭnikko G M and Veretennikov A Yu 1986 *Iteration Procedures in Ill-Posed Problems* (Moscow: Nauka) (in Russian)