

ON ℓ^1 -REGULARIZATION IN LIGHT OF NASHED'S ILL-POSEDNESS CONCEPT

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ABSTRACT. Based on the powerful tool of variational inequalities, in recent papers convergence rates results on ℓ^1 -regularization for ill-posed inverse problems have been formulated in infinite dimensional spaces under the condition that the sparsity assumption slightly fails, but the solution is still in ℓ^1 . In the present paper we improve those convergence rates results and apply them to the Cesáro operator equation in ℓ^2 and to specific denoising problems. Moreover, we formulate in this context relationships between Nashed's types of ill-posedness and mapping properties like compactness and strict singularity.

1. INTRODUCTION

Variational sparsity regularization based on ℓ^1 -norms became of significant interest in the past ten years with respect to inverse problems applications, e.g. in imaging (cf., e.g., [28]), but also with respect to the progress in regularization theory for the treatment of ill-posed operator equations in infinite dimensional Hilbert and Banach spaces (cf., e.g., [3, 7, 15, 16, 23, 27]). Moreover, with focus on sparsity, the use of ℓ^1 -regularization can be motivated for specific classes of well-posed problems, too (cf., e.g., [10]). Based on the powerful tool of variational inequalities (also called variational source conditions), in [9] convergence rates results on ℓ^1 -regularization for linear ill-posed operator equations have been formulated in infinite dimensional spaces under the condition that the sparsity assumption slightly fails, but the solution is still in ℓ^1 . In the present paper, we improve those results and illustrate the improvement level with respect to the associated convergence rates for the Cesáro operator equation in ℓ^2 and for specific denoising problems. Since the variational inequality approach requires injectivity of the forward operator (cf. [9, Proposition 5.6]), we restrict all considerations in this paper to *injective* linear forward operators which also ensure uniquely determined solutions for the corresponding linear operator equations. The focus on injectivity is also motivated by the fact that the ill-posedness concept of Section 4 suggested by M. Z. Nashed (cf. [26]) would require substantial technical refinements in a general Banach space setting if non-injective operators were included.

Let $\tilde{A} : X \rightarrow Y$ be an *injective* and *bounded linear operator* mapping between an *infinite dimensional separable Hilbert space* X and an *infinite dimensional Banach space* Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We are searching for the uniquely determined solution $\tilde{x}^\dagger \in X$ of the linear operator equation

$$(1.1) \quad \tilde{A}\tilde{x} = y, \quad \tilde{x} \in X, \quad y \in \mathcal{R}(\tilde{A}),$$

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where $\mathcal{R}(\tilde{A})$ denotes the range of \tilde{A} . Typically, instead of y only noisy data $y^\delta \in Y$ are available. In this context, we consider the deterministic noise model

$$(1.2) \quad \|y^\delta - y\|_Y \leq \delta$$

with given noise level $\delta > 0$.

If the operator \tilde{A} is *normally solvable*, i.e. its range is a closed subset in Y , then solving the equation (1.1) is a *well-posed problem*. Consequently, for injective \tilde{A} the inverse $\tilde{A}^{-1} : \mathcal{R}(\tilde{A}) \subset Y \rightarrow X$ exists and is also a bounded linear operator. If, on the other hand, the range of \tilde{A} is not closed, the inverse \tilde{A}^{-1} is an unbounded linear operator and solving the equation (1.1) is an *ill-posed problem*. This means that small perturbations in the right-hand side may lead to arbitrarily large error in the solution. Then regularization methods are required for obtaining stable approximate solutions to equation (1.1).

As usual we consider in the sequel the Banach spaces ℓ^q , $1 \leq q < \infty$, and ℓ^∞ of infinite sequences of real numbers with finite norms

$$\|x\|_{\ell^q} := \left(\sum_{k=1}^{\infty} |x_k|^q \right)^{1/q} \quad \text{and} \quad \|x\|_{\ell^\infty} := \sup_{k \in \mathbb{N}} |x_k|.$$

The Banach space c_0 consists of the real sequences $(x_k)_{k \in \mathbb{N}}$ with $\lim_{k \rightarrow \infty} |x_k| = 0$ and is also equipped with the norm $\|x\|_{c_0} := \sup_{k \in \mathbb{N}} |x_k|$. Moreover, by ℓ^0 we denote the set of sparse sequences, where a sequence is *sparse* if only a finite number of components is not zero. For such sequences the number of nonzero components is given by

$$\|x\|_{\ell^0} := \sum_{k=1}^{\infty} |\operatorname{sgn}(x_k)| \quad \text{with} \quad \operatorname{sgn}(t) := \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

Throughout this paper we fix an orthonormal basis $\{u^{(k)}\}_{k \in \mathbb{N}}$ in the Hilbert space X . By $x = (x_k)_{k \in \mathbb{N}} \in \ell^2$ we denote the infinite sequence of corresponding Fourier coefficients of \tilde{x} , i.e.

$$\tilde{x} = \sum_{k=1}^{\infty} x_k u^{(k)}.$$

The synthesis operator $L : \ell^1 \rightarrow X$ defined as $Lx := \tilde{x}$ is an injective bounded linear operator. Note that this operator can be extended to ℓ^2 , but in our setting we define it only on ℓ^1 .

The focus of our studies is on *almost sparse* solutions \tilde{x}^\dagger to equation (1.1). This means that only a finite number of coefficients x_k^\dagger from the infinite sequence $x^\dagger = (x_k^\dagger)_{k \in \mathbb{N}}$ is relevant. Here we do not require strict sparsity, $x^\dagger \in \ell^0$, but we allow an infinite number of nonzero coefficients if they decay fast enough. Precisely, we assume $x^\dagger \in \ell^1$ throughout this paper.

Introducing the operator $A := \tilde{A} \circ L : \ell^1 \rightarrow Y$ our goal is to recover the solution $x^\dagger \in \ell^1$ of

$$(1.3) \quad Ax = y, \quad x \in \ell^1, \quad y \in \mathcal{R}(A),$$

from noisy data $y^\delta \in Y$ satisfying (1.2). The following proposition shows that solving this equation is always an ill-posed problem, even if the original equation (1.1) is well-posed.

For the proof and for further reference we note that the synthesis operator L is a composition $L = U \circ \mathcal{E}_2$ of the embedding operator $\mathcal{E}_2 : \ell^1 \rightarrow \ell^2$ and the Riesz isomorphism $U : \ell^2 \rightarrow X$. Thus, the operator A can be written as a composition

$$(1.4) \quad A = \tilde{A} \circ U \circ \mathcal{E}_2$$

of three injective bounded linear operators.

Proposition 1.1. *The range of A is not closed.*

Proof. Assume that $\mathcal{R}(A)$ is closed. The full preimage of $\mathcal{R}(A)$ with respect to \tilde{A} is $\mathcal{R}(L)$. Thus, L has closed range, too. Looking at the composition (1.4), the full preimage of $\mathcal{R}(L)$ with respect to U is $\mathcal{R}(\mathcal{E}_2) = \ell^1$. Consequently, ℓ^1 would be a closed subspace of ℓ^2 . Since ℓ^1 is dense in ℓ^2 this yields the contradiction $\ell^1 = \ell^2$. \square

The proposition shows that equation (1.3) requires regularization in order to obtain stable approximate solutions. For this purpose we use a variant of variational regularization, called ℓ^1 -regularization, where regularized solutions, denoted by x_α^δ , are minimizers of the extremal problem

$$(1.5) \quad \frac{1}{p} \|Ax - y^\delta\|_Y^p + \alpha \|x\|_{\ell^1} \rightarrow \min, \quad \text{subject to } x \in \ell^1.$$

Here, $1 < p < \infty$ is some exponent and $\alpha > 0$ is a regularization parameter. This regularization parameter is chosen in an appropriate manner, a priori as $\alpha = \alpha(\delta)$ depending on the noise level δ , or a posteriori as $\alpha = \alpha(\delta, y^\delta)$ depending also on the present regularized solution y^δ (for details see Sections 2 and 3 below).

We are interested in error estimates

$$(1.6) \quad \|x_\alpha^\delta - x^\dagger\|_{\ell^1} \leq C_{x^\dagger} \varphi(\delta) \quad \text{for all } 0 < \delta \leq \bar{\delta},$$

where the positive constant C_{x^\dagger} may depend on the solution x^\dagger but not on the noise level $\delta > 0$. The estimates (1.6) can be interpreted as convergence rates

$$(1.7) \quad \|x_\alpha^\delta - x^\dagger\|_{\ell^1} = O(\varphi(\delta)) \quad \text{as } \delta \rightarrow 0$$

with rate functions φ which are concave index functions. Following [24] and [20] we call $\varphi : (0, \infty) \rightarrow (0, \infty)$ an index function if it is a continuous and strictly increasing function with $\lim_{t \rightarrow +0} \varphi(t) = 0$.

The article is organized as follows: in the next section we briefly summarize results on existence, stability and convergence of ℓ^1 -regularized solutions. Section 3 contains the main theorem of this paper which improves the result of Theorem 5.2 from [9] and can lead to better convergence rates. Moreover, Section 4 provides some insight into the interplay between Nashed's ill-posedness concept and mapping properties of the forward operator like compactness and strict singularity. In the final Section 5 we apply our findings to a problem of denoising type.

2. EXISTENCE, STABILITY AND CONVERGENCE OF ℓ^1 -REGULARIZED SOLUTIONS

From the general theory of Tikhonov regularization (cf., e.g., [19, Section 3], [29, Sections 4.1.1 and 4.1.2] and [22, Section 3.1]) one can infer the existence and stability of ℓ^1 -regularized solutions x_α^δ as well as its convergence for $\delta \rightarrow 0$ to the uniquely determined solution x^\dagger of equation (1.3) for appropriate choices of the regularization parameter $\alpha > 0$. For this purpose we summarize the results of Proposition 2.8 and Remark 2.9 from [9] in the following proposition taking

into account that the separable Banach space c_0 is a predual space of ℓ^1 , that the operator $A : \ell^1 \rightarrow Y$ is sequentially weak*-to-weak continuous, and that ℓ^1 satisfies the weak* Kadec-Klee property (for a proof see, e.g., [5, Lemma 2.2]). The weak*-to-weak continuity of A can be shown along the lines of the proof of Lemma 2.7 in [9] when $\mathcal{R}(A^*) \subset c_0$ is valid for the adjoint operator $A^* : Y^* \rightarrow \ell^\infty$ to A . In our setting, $\mathcal{R}(A^*) \subset c_0$ is a consequence of the weak convergence $\tilde{A}u^{(k)} \rightharpoonup 0$ in Y for the prescribed orthonormal system $\{u^{(k)}\}_{k \in \mathbb{N}}$ in the Hilbert space X , which implies the weak convergence of formula (3.11) in Remark 3.6 below such that the corresponding part of the proof of [9, Proposition 2.4] applies even if (3.12) is violated.

Proposition 2.1. *For all $1 < p < \infty$, $\alpha > 0$ and $y^\delta \in Y$ there exist uniquely determined minimizers $x_\alpha^\delta \in \ell^1$ of the extremal problem (1.5). These ℓ^1 -regularized solutions are always sparse, i.e. they satisfy*

$$x_\alpha^\delta \in \ell^0.$$

Furthermore, they are always stable with respect to the data, i.e., small perturbations in y^δ in the norm topology of Y lead only to small changes in x_α^δ with respect to the ℓ^1 -norm.

If $\delta_n \rightarrow 0$ and if the regularization parameters $\alpha_n = \alpha(\delta_n, y^{\delta_n})$ are chosen such that

$$\alpha_n \rightarrow 0 \quad \text{and} \quad \frac{\delta_n^p}{\alpha_n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

then

$$(2.1) \quad \lim_{n \rightarrow \infty} \|x_{\alpha_n}^{\delta_n} - x^\dagger\|_{\ell^1} = 0.$$

3. IMPROVED CONVERGENCE RATES

The norm convergence (2.1) can be arbitrarily slow. In order to obtain convergence rates, also for the ℓ^1 -regularization with regularized solutions x_α^δ defined as minimizers to problem (1.5), a link condition between the smoothness of the solution x^\dagger to (1.3) and the forward operator A is required. From the studies and results of the recent paper [9] we immediately derive the following theorem, where this link condition is a range condition imposed on all unit sequences $e^{(k)} = (0, \dots, 0, 1, 0, \dots)$, $k \in \mathbb{N}$, with respect to the adjoint operator A^* . We mention here that $\{e^{(k)}\}_{k \in \mathbb{N}}$ represents a Schauder basis in the Banach spaces ℓ^q for all $1 \leq q < \infty$.

Theorem 3.1. *Let the operator $A : \ell^1 \rightarrow Y$ from equation (1.3) be such that there exist elements $f^{(k)} \in Y^*$, $k \in \mathbb{N}$, satisfying the range conditions*

$$(3.1) \quad e^{(k)} = A^* f^{(k)}.$$

Then a variational inequality

$$(3.2) \quad \|x - x^\dagger\|_{\ell^1} \leq \|x\|_{\ell^1} - \|x^\dagger\|_{\ell^1} + \varphi_1(\|Ax - Ax^\dagger\|_Y) \quad \text{for all } x \in \ell^1$$

is valid for the concave index function

$$(3.3) \quad \varphi_1(t) = 2 \inf_{n \in \mathbb{N}} \left(\sum_{k=n+1}^{\infty} |x_k^\dagger| + t \sum_{k=1}^n \|f^{(k)}\|_{Y^*} \right).$$

This yields the convergence rate

$$(3.4) \quad \|x_\alpha^\delta - x^\dagger\|_{\ell^1} = O(\varphi_1(\delta)) \quad \text{as} \quad \delta \rightarrow 0$$

for ℓ^1 -regularized solutions x_α^δ and for the uniquely determined solution $x^\dagger \in \ell^1$ of equation (1.3) provided that the regularization parameter $\alpha = \alpha(\delta, y^\delta)$ is chosen appropriately, e.g. according to the discrepancy principle

$$(3.5) \quad \tau_1 \delta \leq \|Ax_{\alpha(\delta, y^\delta)}^\delta - y^\delta\|_Y \leq \tau_2 \delta$$

for prescribed values $1 < \tau_1 \leq \tau_2 < \infty$.

Remark 3.2. For more details concerning the consequences of variational inequalities and the role of the choice of the regularization parameter for obtaining convergence rates in regularization we refer, for example, to [19] and [1, 4, 5, 12, 15, 21]. Making use of Gelfand triples it was shown in [2] that, for a wide range of applied inverse problems, the forward operators A are such that link conditions of the form (3.1) apply for all $e^{(k)}$, $k \in \mathbb{N}$. On the other hand, the paper [13] gives counterexamples where (3.1) fails for specific operators A , but alternative link conditions presented there can compensate this deficit.

We improve the convergence rate obtained in the theorem above as follows:

Theorem 3.3. *Theorem 3.1 remains true if φ_1 is replaced by φ_2 with*

$$(3.6) \quad \varphi_2(t) = 2 \inf_{n \in \mathbb{N}} \left(\sum_{k=n+1}^{\infty} |x_k^\dagger| + t \sup_{\substack{a_k \in \{-1, 0, 1\} \\ k=1, \dots, n}} \left\| \sum_{k=1}^n a_k f^{(k)} \right\|_{Y^*} \right).$$

Proof. From [9, Lemma 5.1] we know that

$$(3.7) \quad \|x - x^\dagger\|_{\ell^1} - \|x\|_{\ell^1} + \|x^\dagger\|_{\ell^1} \leq 2 \left(\sum_{k=n+1}^{\infty} |x_k^\dagger| + \sum_{k=1}^n |x_k - x_k^\dagger| \right).$$

Observing

$$\begin{aligned} |x_k - x_k^\dagger| &= (\operatorname{sgn}(x_k - x_k^\dagger))(x_k - x_k^\dagger) \\ &= (\operatorname{sgn}(x_k - x_k^\dagger)) \langle e^{(k)}, x - x^\dagger \rangle_{\ell^\infty \times \ell^1} \\ &= (\operatorname{sgn}(x_k - x_k^\dagger)) \langle f^{(k)}, Ax - Ax^\dagger \rangle_{Y^* \times Y} \end{aligned}$$

the second sum on the right-hand side can be estimated above by

$$\begin{aligned} \sum_{k=1}^n |x_k - x_k^\dagger| &= \sum_{k=1}^n (\operatorname{sgn}(x_k - x_k^\dagger)) \langle f^{(k)}, Ax - Ax^\dagger \rangle_{Y^* \times Y} \\ &= \left\langle \sum_{k=1}^n (\operatorname{sgn}(x_k - x_k^\dagger)) f^{(k)}, Ax - Ax^\dagger \right\rangle_{Y^* \times Y} \\ (3.8) \quad &\leq \left\| \sum_{k=1}^n (\operatorname{sgn}(x_k - x_k^\dagger)) f^{(k)} \right\|_{Y^*} \|Ax - Ax^\dagger\|_Y \\ &\leq \|Ax - Ax^\dagger\|_Y \sup_{\substack{a_k \in \{-1, 0, 1\} \\ k=1, \dots, n}} \left\| \sum_{k=1}^n a_k f^{(k)} \right\|_{Y^*} \end{aligned}$$

and taking the infimum over all $n \in \mathbb{N}$ in the resulting inequality yields the variational inequality (3.2). \square

To understand the difference between φ_1 and φ_2 it may be helpful to note that $\varphi_1 = \varphi_2$ if all $f^{(k)}$ are pairwise collinear. On the other hand, in the particular case that $Y^* = Y = \ell^2$ and $f^{(k)} = e^{(k)}$, we have $\sum_{k=1}^n \|f^{(k)}\|_{Y^*} = n$ while $\sup_{a_k \in \{-1,0,1\}} \left\| \sum_{k=1}^n a_k f^{(k)} \right\|_{Y^*} = \sqrt{n}$.

In Example 3.7 and in Section 5 we will show that the improved index function φ_2 yields better convergence rates for some equations than the original function φ_1 . The convergence rate result obtained in [13] can be improved in a similar way.

Example 3.4 (Hölder rates). If

$$(3.9) \quad \sum_{k=n+1}^{\infty} |x_k^\dagger| \leq K_1 n^{-\mu} \quad \text{and} \quad \sup_{a_k \in \{-1,0,1\}} \left\| \sum_{k=1}^n a_k f^{(k)} \right\|_{Y^*} \leq K_2 n^\nu$$

for $K_1, K_2 \geq 0$ and $\mu, \nu > 0$, then the convergence rate obtained in Theorem 3.3 is

$$(3.10) \quad \|x_\alpha^\delta - x^\dagger\|_{\ell^1} = O(\delta^{\frac{\mu}{\mu+\nu}}) \quad \text{as} \quad \delta \rightarrow 0$$

(cf. [9, Example 5.3]).

Example 3.5 (exponential decay of solution components). If

$$\sum_{k=n+1}^{\infty} |x_k^\dagger| \leq K_1 \exp(-n^\gamma) \quad \text{and} \quad \sup_{a_k \in \{-1,0,1\}} \left\| \sum_{k=1}^n a_k f^{(k)} \right\|_{Y^*} \leq K_2 n^\nu$$

for $K_1, K_2 \geq 0$ and $\gamma, \nu > 0$, then the convergence rate obtained in Theorem 3.3 is

$$\|x_\alpha^\delta - x^\dagger\|_{\ell^1} = O\left(\delta(\log(1/\delta))^{\frac{\nu}{\gamma}}\right) \quad \text{as} \quad \delta \rightarrow 0$$

(cf. [5, Example 3.5]).

Remark 3.6. Note that we always have weak convergence

$$(3.11) \quad Ae^{(k)} \rightharpoonup 0 \quad \text{in } Y \quad \text{as} \quad k \rightarrow \infty,$$

because $\{e^{(k)}\}_{k \in \mathbb{N}}$ converges weakly in ℓ^2 and \tilde{A} is weak-to-weak continuous since it is norm-to-norm continuous. In [9, Remark 2.5] it was shown that the slightly stronger condition

$$(3.12) \quad \lim_{k \rightarrow \infty} \|Ae^{(k)}\|_Y = 0.$$

enforces $\|f^{(k)}\|_{Y^*} \rightarrow \infty$ in Theorems 3.1 and 3.3.

Obviously, condition (3.12) is satisfied if the underlying operator \tilde{A} is compact since compact operators map weakly convergent sequences to norm convergent ones (note that this property is equivalent to compactness of \tilde{A} if X is a Hilbert or at least a reflexive Banach space, cf. [25, Thm. 3.4.37]). On the other hand, one easily finds examples for noncompact operators which do not satisfy (3.12). Choose, e.g., $X = Y = \ell^2$ and let \tilde{A} be the identity. Then $\|Ae^{(k)}\|_Y = 1$ for all $k \in \mathbb{N}$. The question arises whether (3.12) is equivalent to compactness of \tilde{A} . The answer is ‘no’ as the following example demonstrates.

Example 3.7 (Cesàro operator). Let $X = Y = \ell^2$ and define $\tilde{A} : \ell^2 \rightarrow \ell^2$ by

$$(3.13) \quad [\tilde{A}x]_n = \frac{1}{n} \sum_{k=1}^n x_k.$$

This operator is injective and noncompact with nonclosed range (see [17, Solution 177] or [8]), but we have

$$\|Ae^{(k)}\|_{\ell^2}^2 = \sum_{n=k}^{\infty} \frac{1}{n^2} \rightarrow 0 \quad \text{if } k \rightarrow \infty.$$

Since with $f^{(1)} := e^{(1)}$ and $f^{(k)} := ke^{(k)} - (k-1)e^{(k-1)}$ for $k \geq 2$ assumption (3.1) of Theorems 3.1 and 3.3 is satisfied, both convergence rates results apply to the specified operator.

In the index function φ_1 in Theorem 3.1 the second sum is

$$\sum_{k=1}^n \|f^{(k)}\|_{\ell^2} = \sum_{k=1}^n \sqrt{(k-1)^2 + k^2} \leq \frac{n(n+1)}{\sqrt{2}}.$$

From

$$\sqrt{(k-1)^2 + k^2} \geq \frac{1}{\sqrt{2}}(k-1+k) = \sqrt{2}k - \frac{1}{\sqrt{2}}$$

we even obtain a lower bound of the same order

$$\sum_{k=1}^n \|f^{(k)}\|_{\ell^2} \geq \frac{\sqrt{2}n(n+1)}{2} - \frac{n}{\sqrt{2}} = \frac{n^2}{\sqrt{2}}.$$

On the other hand we now show that the supremum in the definition of φ_2 can be estimated above by

$$\sup_{\substack{a_k \in \{-1, 0, 1\} \\ k=1, \dots, n}} \left\| \sum_{k=1}^n a_k f^{(k)} \right\|_{\ell^2} \leq \frac{2}{\sqrt{3}} n^{3/2}.$$

At first we calculate

$$\begin{aligned} \sum_{k=1}^n a_k f^{(k)} &= a_1 e^{(1)} + \sum_{k=2}^n k a_k e^{(k)} - \sum_{k=1}^{n-1} k a_{k+1} e^{(k)} \\ &= n a_n e^{(n)} + \sum_{k=1}^{n-1} k (a_k - a_{k+1}) e^{(k)} \end{aligned}$$

for arbitrary a_1, \dots, a_n . Thus,

$$\left\| \sum_{k=1}^n a_k f^{(k)} \right\|_{\ell^2} = \sqrt{n^2 a_n^2 + \sum_{k=1}^{n-1} k^2 (a_k - a_{k+1})^2},$$

which attains its maximum over $(a_1, \dots, a_n) \in \{-1, 0, 1\}^n$ for $a_k = (-1)^k$. Then

$$\sup_{\substack{a_k \in \{-1, 0, 1\} \\ k=1, \dots, n}} \left\| \sum_{k=1}^n a_k f^{(k)} \right\|_{\ell^2} = \sqrt{n^2 + 4 \sum_{k=1}^{n-1} k^2} = \sqrt{\frac{4}{3}n^3 - n^2 + \frac{2}{3}n} \leq \frac{2}{\sqrt{3}} n^{3/2}.$$

Here, Examples 3.4 and 3.5 apply and from these examples we see that the behaviour of the estimated sum in φ_1 and of the supremum in φ_2 directly carries over to the convergence rate. Thus, the slower growth of the supremum in comparison

to the faster growth of the sum yields a better rate for $\nu = \frac{3}{2}$ based on Theorem 3.3 than for $\nu = 2$ based on Theorem 3.1.

Example 3.8 (diagonal operator). For a comparison we briefly recall Example 2.6 from [9], where $\tilde{A} : X \rightarrow Y$ is a *compact diagonal operator* between the separable Hilbert spaces X and Y with the singular system $\{\sigma_k, u^{(k)}, v^{(k)}\}_{k \in \mathbb{N}}$ and $\tilde{A}u^{(k)} = \sigma_k v^{(k)}$, $k \in \mathbb{N}$. Then the decay rate of the singular values $\sigma_k \rightarrow 0$ for $k \rightarrow \infty$ characterizes the degree of ill-posedness of the equation (1.1). For $\sigma_k \sim k^{-\zeta}$, $\zeta > 0$, we have $\|Ae^{(k)}\|_Y = \|\tilde{A}u^{(k)}\|_Y = \sigma_k \sim k^{-\zeta}$. The link condition (3.1) is satisfied with $f^{(k)} = \frac{1}{\sigma_k}v^{(k)}$ and $\|f^{(k)}\|_Y \sim k^\zeta \rightarrow \infty$ as $k \rightarrow \infty$. Moreover, we have with some constant $C > 0$

$$\sup_{\substack{a_k \in \{-1, 0, 1\} \\ k=1, \dots, n}} \left\| \sum_{k=1}^n a_k f^{(k)} \right\|_Y = \sqrt{\sum_{k=1}^n \frac{1}{\sigma_k^2}} \leq \sum_{k=1}^n \frac{1}{\sigma_k} \leq C n^{\zeta+1},$$

hence $\nu = \zeta + 1 > 1$ in Examples 3.4 and 3.5 based on Theorem 3.3. Note that the values $\zeta > \frac{1}{2}$ and consequently $\nu > \frac{3}{2}$ correspond with the case of Hilbert-Schmidt operators \tilde{A} and $\nu = \frac{3}{2}$ occurring in Example 3.7 is just a borderline case with respect to that fact.

4. ILL-POSEDNESS OF TYPE I AND II

As suggested by M. Z. Nashed in [26] we distinguish two types of ill-posedness for linear operator equations in a Banach space setting. Again, our focus is on injective operators.

Definition 4.1. Let $B : Z_1 \rightarrow Z_2$ be an injective and bounded linear operator mapping between the infinite dimensional Banach spaces Z_1 and Z_2 . Then the operator equation

$$(4.1) \quad Bx = y$$

is called *well-posed* if the range $\mathcal{R}(B)$ is a closed subset of Z_2 , consequently ill-posed if the range is not closed, i.e. $\mathcal{R}(B) \neq \overline{\mathcal{R}(B)}^{Z_2}$.

In the ill-posed case, the equation (4.1) is called *ill-posed of type I* if the range $\mathcal{R}(B)$ contains an infinite dimensional closed subspace, and it is called *ill-posed of type II* otherwise.

The two types of ill-posedness differ in the behavior of corresponding regularizers (cf. [26]) and with respect to smoothing properties of the linear operators B . If $B := B_1 : Z_1 \rightarrow Z_2$ is such that equation (4.1) proves to be ill-posed of type I and $B := B_2 : Z_1 \rightarrow Z_2$ is such that equation (4.1) proves to be ill-posed of type II, then B_2 tends to be ‘more smoothing’ than B_1 . Namely, a range inclusion $\mathcal{R}(B_2) \subset \mathcal{R}(B_1)$ may occur, but $\mathcal{R}(B_1) \subset \mathcal{R}(B_2)$ cannot apply. We refer to [6] for consequences of range inclusions and in particular to Example 10.2 ibidem for the interplay of operators which characterize different types of ill-posedness.

The following proposition shows that at least for operators B between Hilbert spaces Z_1 and Z_2 the type of ill-posedness is determined by compactness properties of B .

Proposition 4.2. *If the operator equation (4.1) is well-posed or ill-posed of type I, then the operator B is non-compact. Consequently, compactness of B implies ill-posedness of type II.*

If Z_1 and Z_2 in equation (4.1) are Hilbert spaces and the equation is ill-posed, then the equation is ill-posed of type II if and only if B is compact.

Proof. If the operator equation (4.1) is well-posed or ill-posed of type I, there is an infinite dimensional Banach space \hat{Z}_2 included in the subspace $\mathcal{R}(B)$ of Z_2 with the same norm as in Z_2 . The preimage $\hat{Z}_1 := B^{-1}(\hat{Z}_2)$ is a Banach space included in Z_1 with the same norm as in Z_1 . For a compact operator B , also its restriction $B|_{\hat{Z}_1} : \hat{Z}_1 \rightarrow \hat{Z}_2$ would be compact and moreover surjective. This contradicts the fact that a compact operator has only a closed range if it has a finite dimensional range, being a consequence of the non-compactness of the unit ball in a infinite dimensional Banach space.

For Hilbert spaces Z_1 and Z_2 and ill-posed equations (4.1), the equivalence of ill-posedness of type I and the non-compactness of B is well-known (cf. [26, Thm. 4.6] and [11, Lemma 5.8 and Theorem 5.9]). \square

If compactness of B is replaced by strict singularity, the characterization of ill-posedness types can be made more precise for injective operators B .

Definition 4.3. *A bounded linear operator B between Banach spaces Z_1 and Z_2 is strictly singular if its restriction to an infinite dimensional subspace is never an isomorphism.*

Proposition 4.4. *Let B be an injective bounded linear operator between Banach spaces Z_1 and Z_2 . Then equation (4.1) is ill-posed of type II if and only if B is strictly singular.*

Proof. We show that there exists an isomorphic restriction of B to an infinite dimensional subspace of Z_1 if and only if $\mathcal{R}(B)$ contains a closed infinite dimensional subspace.

Obviously, if Z_3 is a closed infinite dimensional subspace of $\mathcal{R}(B)$, then the corresponding preimage is of infinite dimension and the restriction of B to this preimage is an isomorphism. On the other hand, if there is an isomorphic restriction to an infinite dimensional subspace of Z_1 , then its image is also of infinite dimension and closed. \square

Now we are going to apply Definition 4.1 to the equations (1.1) and (1.3) and to interpret the different cases. First we distinguish in the subsequent remark the possible cases arising in the context of equation (1.1).

Remark 4.5.

- (a) **Well-posed case:** The equation (1.1) can be well-posed, which takes place if \tilde{A} is normally solvable. Linear Volterra integral equations of the second kind as well as more generally linear Fredholm integral equations of the second kind with appropriate kernels represent typical examples of this case, where $X = Y = L^2(\Omega)$ with some bounded and sufficiently regular domain Ω in \mathbb{R}^l , $l = 1, 2, \dots$, and the operator \tilde{A} is of the form $\tilde{A} = I - K$ with the identity operator I and a compact operator K such that zero does not belong to the spectrum of the operator \tilde{A} . Normal solvability

also occurs if $X = Y$ and $\tilde{A} = I$. Then solving (1.1), for given noisy data $y^\delta \in Y$, is the simplest case of a *denoising* problem (cf. Section 5).

- (b) **Ill-posed case:** The equation (1.1) is ill-posed if the operator \tilde{A} fails to be normally solvable. This is just the case if the inverse $\tilde{A}^{-1} : \mathcal{R}(\tilde{A}) \subset Y \rightarrow X$ is unbounded.

- (b1) **Type I:** Operators \tilde{A} for equations (1.1) which prove to be ill-posed of type I are non-compact and even not strictly singular. If Y is also a Hilbert space, all ill-posed equations (1.1) with non-compact operator \tilde{A} are of this type. Multiplication operators in $X = Y = L^2(a, b)$ with $L^\infty(a, b)$ -multiplier functions possessing essential zeros and linear convolution operators in $X = Y = L^2(\mathbb{R}^l)$ with square-integrable kernels represent typical examples for this case. Furthermore, the Hausdorff moment problem with $\tilde{A} : L^2(0, 1) \rightarrow \ell^2$ (cf. [18, Example 3.2]) is of this type.

- (b2) **Type II, non-compact:** If Y is not a Hilbert space then there exist strictly singular operators \tilde{A} with nonclosed range which are not compact, for example the embedding operators from ℓ^2 to ℓ^q with $2 < q < \infty$ (cf. [14, Theorem (a)]). This also leads to ill-posed equations (1.1) of type II. Such operators \tilde{A} can have a non-separable range $\mathcal{R}(\tilde{A})$ (cf. [14, Remark on p. 335]).

- (b3) **Type II, compact:** The equation (1.1) is ill-posed of type II if $\tilde{A} : X \rightarrow Y$ is a *compact* operator. Then \tilde{A} is strictly singular and the range $\mathcal{R}(\tilde{A})$ is a separable space. Typical examples with compact operators \tilde{A} are linear Fredholm and Volterra integral *equations of the first kind* with square-integrable kernels in L^2 -spaces X and Y over bounded and sufficiently regular domains in \mathbb{R}^l . Moreover, all bounded linear operators $\tilde{A} : \ell^2 \rightarrow \ell^q$ are compact for $1 \leq q < 2$ as was mentioned in [30]. In particular, if also Y is a Hilbert space, then vice versa ill-posedness of type II requires compactness of \tilde{A} .

The diagram in Figure 1 illustrates the different cases in Remark 4.5 and the results of this section concerning the relations between compactness, strict singularity and type of ill-posedness for injective forward operators. By the way, we should mention that there exist non-injective strictly singular operators possessing a range which contains an infinite dimensional closed subspace (cf. [14, first example]).

A scenario completely different from Remark 4.5 occurs for equation (1.3) due to the composition structure (1.4) of the operator A . The fact that the non-compact embedding operator \mathcal{E}_2 is strictly singular (cf. [14, Theorem]) prevents the occurrence of well-posedness and ill-posedness of type I in the context of this equation.

Proposition 4.6. *Under the assumptions stated above equation (1.3) is always ill-posed of type II.*

Proof. Taking into account Proposition 4.4 we only have to show that A is always strictly singular. But this follows immediately from the composition structure (1.4) and the two facts that \mathcal{E}_2 is strictly singular (see [14]) and that the composition of a strictly singular operator with a bounded linear operator is again strictly singular. \square

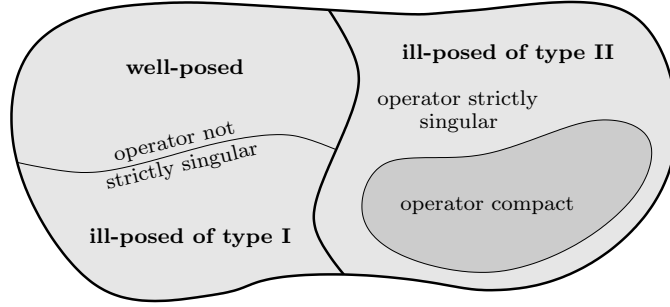


FIGURE 1. Relations between strict singularity, compactness and type of ill-posedness for equations (4.1) with injective bounded linear operator.

5. THE SPECIAL CASE OF DENOISING

Finally, we apply our results to a typical denoising problem. Given a noisy signal one wants to remove the noise. For this purpose one decomposes the signal with respect to a wavelet basis (or any other orthonormal system) and tries to find a sparse approximation with respect to this basis. Thus, in our setting we choose \tilde{A} to be the identity on ℓ^2 . Since in some applications it might be reasonable to measure the noise in a weaker norm we extend Y to ℓ^q with $2 \leq q \leq \infty$. Then $A := \mathcal{E}_q$ is the embedding of ℓ^1 into ℓ^q . In the sequel we only look at A and therefore extend the feasible values for q to $1 \leq q \leq \infty$.

The minimization problem (1.5) now reads as

$$(5.1) \quad \frac{1}{p} \|\mathcal{E}_q x - y^\delta\|_{\ell^q}^p + \alpha \|x\|_{\ell^1} \rightarrow \min, \quad \text{subject to } x \in \ell^1,$$

where the exact signal x^\dagger is assumed to be nearly sparse, i.e. $x^\dagger \in \ell^1$. From the computational point of view it seems to be helpful to apply $p := q$ for the exponent in the misfit term of (5.1) whenever $1 < q < \infty$. If we measure the error after denoising in the ℓ^1 -norm as $\|x_\alpha^\delta - x^\dagger\|_{\ell^1}$, then the chances of having small errors improve with decreasing values q , since the strength of the norm in Y grows if q decreases.

Proposition 5.1. *For the embedding operator $A = \mathcal{E}_q$ from ℓ^1 to ℓ^q with $1 < q \leq \infty$ equation (1.3) is ill-posed of type II. We have weak convergence $Ae^{(k)} \rightharpoonup 0$ if $k \rightarrow \infty$ for $1 < q \leq \infty$, but no convergence in norm. For all $1 \leq q \leq \infty$ and all $k \in \mathbb{N}$ the link condition (3.1) is satisfied with $f^{(k)} = e^{(k)}$. Thus, Theorems 3.1 and 3.3 apply and the corresponding index functions are*

$$(5.2) \quad \varphi_1(t) = 2 \inf_{n \in \mathbb{N}} \left(\sum_{k=n+1}^{\infty} |x_k^\dagger| + t n \right) \quad \text{if } 1 \leq q \leq \infty$$

and

$$(5.3) \quad \varphi_2(t) = 2 \inf_{n \in \mathbb{N}} \left(\sum_{k=n+1}^{\infty} |x_k^\dagger| + t n^\theta \right)$$

with

$$\theta = \begin{cases} 1 - \frac{1}{q}, & \text{if } 1 \leq q < \infty, \\ 1, & \text{if } q = \infty. \end{cases}$$

Proof. The ill-posedness of type II is a consequence of Proposition 4.6 whenever $2 \leq q \leq \infty$, because the injective and bounded embedding operator from ℓ^2 to ℓ^q plays here the role of \tilde{A} . On the other hand, the non-existence of an infinite dimensional closed subspace in ℓ^q for $1 < q < 2$ included in ℓ^1 also follows from the theorem in [14], since the corresponding embedding operators are strictly singular.

It is evident that $e^{(k)}$ does not converge to zero in the norm of ℓ^q , but we have $e^{(k)} \rightarrow 0$ in ℓ^q for all $1 < q \leq \infty$. Also the validity of (3.1) is evident.

The index functions φ_1 and φ_2 in the convergence rates theorems can be computed easily for the special case under consideration. \square

Example 5.2 (Hölder rates). If the decay rate $x_k^\dagger \rightarrow 0$ as $k \rightarrow \infty$ of the remaining solution coefficients is of power type

$$(5.4) \quad |x_k^\dagger| \leq C_{x^\dagger} k^{\mu-1}, \quad k \in \mathbb{N}, \quad \text{or equivalently} \quad \sum_{k=n+1}^{\infty} |x_k^\dagger| \leq K_{x^\dagger} n^{-\mu}, \quad n \in \mathbb{N},$$

with constants $\mu > 0$ and $C_{x^\dagger}, K_{x^\dagger} > 0$, we immediately derive from Proposition 5.1 and Example 3.4 the Hölder convergence rates for the denoising problem with forward operator $A = \mathcal{E}_q$ as

$$(5.5) \quad \|x_\alpha^\delta - x^\dagger\|_{\ell^1} = O\left(\delta^{\frac{\mu}{\mu+1-\frac{1}{q}}}\right) \quad \text{as } \delta \rightarrow 0 \quad \text{if } 1 \leq q < \infty$$

and

$$(5.6) \quad \|x_\alpha^\delta - x^\dagger\|_{\ell^1} = O\left(\delta^{\frac{\mu}{\mu+1}}\right) \quad \text{as } \delta \rightarrow 0 \quad \text{if } q = +\infty.$$

As expected the rate grows if q decreases, i.e., if the noise is measured in a stronger norm. On the other hand, the borderline case $q = 1$ leads to a well-posed equation (1.3). In this case the index function φ_2 attains the form

$$\varphi_2(t) = 2 \inf_{n \in \mathbb{N}} \left(\sum_{k=n+1}^{\infty} |x_k^\dagger| + t \right) = 2t \quad \text{if } q = 1$$

and the corresponding rate is

$$\|x_\alpha^\delta - x^\dagger\|_{\ell^1} = O(\delta) \quad \text{as } \delta \rightarrow 0 \quad \text{if } q = 1,$$

which is typical for well-posed situations.

The example also shows that the improved index function φ_2 from Theorem 3.3 indeed provides a better convergence rate for $1 \leq q < \infty$ than the original index function φ_1 from Theorem 3.1. Note that φ_1 in Proposition 5.1 is for all q the same function as φ_2 with $q = \infty$.

At the end we should mention that the rate results (5.5) and (5.6) yield the values $0 < \nu \leq 1$ in formula (3.10) from Example 3.4. Consequently, the Hölder rates in Examples 3.7 and 3.8 with $\nu > 1$ are always lower than the observed rates for the denoising case, which indicates a lower degree of ill-posedness for the denoising problem.

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REFERENCES

- [1] S. W. Anzengruber, B. Hofmann, and P. Mathé. Regularization properties of the sequential discrepancy principle for Tikhonov regularization in Banach spaces. *Appl. Anal.*, 93:1382–1400, 2014.
- [2] S. W. Anzengruber, B. Hofmann, and R. Ramlau. On the interplay of basis smoothness and specific range conditions occurring in sparsity regularization. *Inverse Problems*, 29:125002 (21pp), 2013.
- [3] M. Benning and M. Burger. Error estimates for general fidelities. *Electronic Transactions on Numerical Analysis*, 38:44–68, 2011.
- [4] R. I. Boţ and B. Hofmann. An extension of the variational inequality approach for obtaining convergence rates in regularization of nonlinear ill-posed problems. *Journal of Integral Equations and Applications*, 22:369–392, 2010.
- [5] R. I. Boţ and B. Hofmann. The impact of a curious type of smoothness conditions on convergence rates in ℓ^1 -regularization. *Eurasian Journal of Mathematical and Computer Applications*, 1:29–40, 2013.
- [6] A. Böttcher, B. Hofmann, U. Tautenhahn, and M. Yamamoto. Convergence rates for Tikhonov regularization from different kinds of smoothness conditions. *Appl. Anal.*, 85:555–578, 2006.
- [7] K. Bredies and D. A. Lorenz. Regularization with non-convex separable constraints. *Inverse Problems*, 25:085011 (14pp), 2009.
- [8] A. Brown, P. R. Halmos, and A. L. Shields. Cesàro operators. *Acta Sci. Math. (Szeged)*, 26:125–137, 1965.
- [9] M. Burger, J. Flemming, and B. Hofmann. Convergence rates in ℓ^1 -regularization if the sparsity assumption fails. *Inverse Problems*, 29:025013 (16pp), 2013.
- [10] E. J. Candès, J. K. Romberg, and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. *Comm. Pure Appl. Math.*, 59:1207–1223, 2006.
- [11] R. G. Douglas. *Banach Algebra Techniques in Operator Theory*, volume 179 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [12] J. Flemming. *Generalized Tikhonov Regularization and Modern Convergence Rate Theory in Banach Spaces*. Shaker Verlag, Aachen, 2012.
- [13] J. Flemming and M. Hegland. Convergence rates in ℓ^1 -regularization when the basis is not smooth enough. *Appl. Anal.*, 94:464–476, 2015.
- [14] S. Goldberg and E. Thorp. On some open questions concerning strictly singular operators. *Proc. Amer. Math. Soc.*, 14:334–336, 1963.
- [15] M. Grasmair. Generalized Bregman distances and convergence rates for non-convex regularization methods. *Inverse Problems*, 26:115014 (16pp), 2010.
- [16] M. Grasmair, M. Haltmeier, and O. Scherzer. Necessary and sufficient conditions for linear convergence of ℓ^1 -regularization. *Comm. Pure Appl. Math.*, 64:161–182, 2011.
- [17] P. R. Halmos. *A Hilbert Space Problem Book*, volume 19 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1982. Encyclopedia of Mathematics and its Applications, 17.
- [18] B. Hofmann. *Mathematik inverser Probleme*. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1999.
- [19] B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators. *Inverse Problems*, 23:987–1010, 2007.

- [20] B. Hofmann and P. Mathé. Analysis of profile functions for general linear regularization methods. *SIAM J. Numer. Anal.*, 45:1122–1141, 2007.
- [21] B. Hofmann and P. Mathé. Parameter choice in Banach space regularization under variational inequalities. *Inverse Problems*, 28:104006 (17pp), 2012.
- [22] K. Ito and B. Jin. *Inverse Problems - Tikhonov Theory and Algorithms*, volume 22 of *Series on Applied Mathematics*. World Scientific, Singapore, 2014.
- [23] D. A. Lorenz. Convergence rates and source conditions for Tikhonov regularization with sparsity constraints. *J. Inverse Ill-Posed Probl.*, 16:463–478, 2008.
- [24] P. Mathé and S. V. Pereverzev. Geometry of linear ill-posed problems in variable Hilbert scales. *Inverse Problems*, 19:789–803, 2003.
- [25] R. E. Megginson. *An Introduction to Banach Space Theory*, volume 183 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.
- [26] M. Z. Nashed. A new approach to classification and regularization of ill-posed operator equations. In *Inverse and Ill-posed Problems (Sankt Wolfgang, 1986)*, volume 4 of *Notes Rep. Math. Sci. Engrg.*, pages 53–75. Academic Press, Boston, MA, 1987.
- [27] R. Ramlau and E. Resmerita. Convergence rates for regularization with sparsity constraints. *Electron. Trans. Numer. Anal.*, 37:87–104, 2010.
- [28] O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, and F. Lenzen. *Variational Methods in Imaging*, volume 167 of *Applied Mathematical Sciences*. Springer, New York, 2009.
- [29] T. Schuster, B. Kaltenbacher, B. Hofmann, and K. S. Kazimierski. *Regularization Methods in Banach Spaces*, volume 10 of *Radon Series on Computational and Applied Mathematics*. Walter de Gruyter, Berlin/Boston, 2012.
- [30] E. Tarafdar. A note on the bounded linear operators on the spaces l^p and L_p . *J. Math. Anal. Appl.*, 40:683–686, 1972.

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