

Exercises Singularity Theory

1. (1 point)

Show that the following smooth functions (the so-called “elementary catastrophes”) have isolated critical points at 0 in \mathbb{R} resp. in \mathbb{R}^2 and no other critical points.

$$x^3, x^4, x^5, x^6, x^3 + y^3, x^3 - xy^2, x^2y + y^4$$

2. (4 Punkte) A reminder from linear algebra:

Let V be a real vector space of dimension n and $h : V \times V \rightarrow \mathbb{R}$ a symmetric bilinear map. For a given basis (v_1, \dots, v_n) of V , the map h determines a symmetric matrix $H \in \text{Sym}(n, \mathbb{R}) \subset \text{Mat}(n \times n, \mathbb{R})$ by putting $H_{ij} := h(v_i, v_j)$.

The aim of the exercise is to show that there is always a basis \underline{v} , such that the matrix H has a particular simple shape. This is sometimes called “Sylvester’s law of inertia” (Sylvesterscher Trägheitssatz).

First we recall some useful notions needed for the statement and the proof.

- The radical of h is defined as $\text{Rad}(h) := \{v \in V \mid h(v, w) = 0 \ \forall w \in V\}$.
- h is called non-degenerate, if $\text{Rad}(h) = \{0\}$.
- h is called positiv definite resp. negativ definite, if we have $h(v, v) > 0$ resp. $h(v, v) < 0$ for all $v \in V \setminus \{0\}$.

Show Sylvester’s law of inertia using the following hints.

- (a) (1 point) Let U be a subspace of V such that we have $V = \text{Rad}(h) \oplus U$. Show that the restriction of h to U is non-degenerate.
- (b) (1 point) Let h be non-degenerate and $w \in V$ such that $h(w, w) \neq 0$. Let $W := (\mathbb{R}w)^\perp$ be the orthogonal complement of the space generated by w , i.e.

$$W := \{v \in V \mid h(v, w) = 0\}.$$

Show that the restriction of h to W is non-degenerate.

- (c) (2 points) Show by induction on n and using statements (a) and (b) that there is a basis $\underline{v} := (v_1, \dots, v_n)$ of V such that the matrix H representing h with respect to \underline{v} is diagonal of the form $\text{diag}(\lambda_1, \dots, \lambda_k, \epsilon_1, \dots, \epsilon_l, \nu_1, \dots, \nu_r)$, and such that $\lambda_1 = \dots = \lambda_k = 1$, $\epsilon_1 = \dots = \epsilon_l = -1$ und $\nu_1 = \dots = \nu_r = 0$.

Show moreover that the numbers k, l und r are independent of the choice of the basis. They are called Sylvester’s invariants (of h), in particular, the number $n - r$ is called the rank of h , r is called the corank of h and l is called the index of h .

3. (2 points) Let $g \in \mathbb{R}[x]$ be a polynomial, $p \in \mathbb{N}$ and let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) := y^p - g(x)$.

- (a) Show that $\text{Crit}(f) \cap f^{-1}(0) = \{(x, 0) \mid g \text{ has multiple roots at } x\}$.
- (b) Draw the set $f^{-1}(0) \subset \mathbb{R}^2$ in the following cases
- $p = 1, 2$ und $g = x^2 - 1$,
 - $p = 2$ und $g = x^2 \cdot (1 - x^2)^3$.

To be handed in until Wednesday, 15th April 2020